JOURNAL OF COMPUTATIONAL PHYSICS 21, 351–355 (1976)

Note

A Note on the Leap-Frog Scheme in Two and Three Dimensions*

The leap-frog finite-difference method [1] for solving hyperbolic linear and quasi-linear systems of partial differential equations is convenient to program and requires few function evaluations. We consider the following system of equations:

$$\frac{\partial u}{\partial t} = \frac{\partial F}{\partial x} + \frac{\partial G}{\partial y} + \frac{\partial H}{\partial z} \tag{1}$$

where $u(x, y, z; t)$ is a q component-vector function; for example in hydrodynamics u is a five-vector. The vectors F, G, and H are functions only of u, and we denote by A, B, and C their Jacobians with respect to u, i.e., $A = A(u) = \partial F/\partial u$, etc.

The leap-frog finite-difference scheme is the following:

$$\begin{aligned} u^{n+1}_{j,k,m} = u^{n-1}_{j,k,m} &+ (\Delta t/\Delta x)[F(u^n_{j+1,k,m}) - F(u^n_{j-1,k,m})] \\ &+ (\Delta t/\Delta y)[G(u^n_{j,k+1,m}) - G(u^n_{j,k-1,m})] \\ &+ (\Delta t/\Delta z)[H(u^n_{j,k,m+1}) - H(u^n_{j,k,m-1})] \end{aligned} \tag{2}$$

The stability condition for the one-dimensional case, i.e., $G = H = 0$, is

$$\Delta t \leqslant \Delta x/\rho(A) \tag{3}$$

where $\rho(A)$ is the spectral radius of $A(u)$. In two space-dimensions, $H = 0$, the stability condition is

$$\Delta t \leqslant \frac{1}{(\rho(A)/\Delta x) + (\rho(B)/\Delta y)} \tag{4}$$

where it has been assumed that A and B are simultaneously symmetrizable.

Under similar conditions on A, B and C, the three-dimensional stability condition is

$$\Delta t \leqslant \frac{1}{(\rho(A)/\Delta x) + (\rho(B)/\Delta y) + (\rho(C)/\Delta z)}\,. \tag{5}$$

* This paper was prepared as a result of work performed under NASA Grant NGR 47-102-001 and Contract Number NAS1-14101 while the authors were in residence at ICASE, NASA Langley Research Center, Hampton, Virginia 23665.

It should be pointed out that the simultaneous symmetrizability constraints on A, B, and C are not severe in practice. For example, the hydrodynamic equations satisfy this restriction [2]. When $A = B$ and $\Delta x = \Delta y$, (4) gives half the theoretically allowable CFL time step. Similarly the worst case of (5) gives $\frac{1}{3}$ the theoretical allowable CFL time step. We shall now show how by a slight modification of the leap-frog algorithm one may improve on (4) and (5) by factors of 2 and 2.8, respectively.

Consider the two-dimensional case first. We propose to solve (1), with $H = 0$, by the following finite-difference scheme:

$$u_{j,k}^{n+1} = u_{j,k}^{n-1} + (\Delta t/\Delta x)\{F[(1/2)(u_{j+1,k+1}^n + u_{j+1,k-1}^n)] - F[(1/2)(u_{j+1,k+1}^n + u_{j-1,k-1}^n)]\}$$
$$+ (\Delta t/\Delta y)\{G[(1/2)(u_{j+1,k+1}^n + u_{j-1,k+1}^n)] - G[(1/2)(u_{j+1,k-1}^n + u_{j-1,k-1}^n)]\}. \quad (6)$$

Standard stability analysis [3] leads to the requirement that the absolute magnitude of the eigenvalues of the matrix

$$D = 2\tilde{A}\xi(1 - \xi^2)^{1/2}(1 - 2\eta^2) + 2\tilde{B}\eta(1 - \eta^2)^{1/2}(1 - 2\xi^2) \quad (7)$$

be less than or equal to unity, where

$$\tilde{A} = (\Delta t/\Delta x)A, \qquad \tilde{B} = (\Delta t/\Delta y)B \quad (8)$$

and $\xi = \sin(\alpha/2)$ and $\eta = \sin(\beta/2)$ where α and β are the two fourier transform variables. Due to the symmetry of A and B we may, in view of (7), write

$$\rho(D) \leqslant 2\rho(\tilde{A})|\,\xi\,|(1 - \xi^2)^{1/2}\,|(1 - 2\eta^2)| + 2\rho(\tilde{B})|\,\eta(1 - \eta^2)^{1/2}\,|(1 - 2\xi^2)| \quad (9)$$

and hence a sufficient condition for stability is

$$\rho(\tilde{A})|\,\xi\,||\,1 - 2\eta^2\,|(1 - \xi^2)^{1/2} + \rho(\tilde{B})|\,\eta\,||\,1 - 2\xi^2\,|(1 - \eta^2)^{1/2} \leqslant \tfrac{1}{2}. \quad (10)$$

It is clear from (10) that the best that we could hope to achieve is $\rho(\tilde{A}) \leqslant 1$, $\rho(\tilde{B}) \leqslant 1$. We shall now show that this is also a sufficient condition, by proving that

$$(\xi^2(1 - \xi^2)(1 - 2\eta^2)^2)^{1/2} + (\eta^2(1 - \eta^2)(1 - 2\xi^2)^2)^{1/2} \leqslant \tfrac{1}{2}. \quad (11)$$

We rewrite the left-hand side of (11) to get

$$2(\tfrac{1}{4} - (\xi^2 - \tfrac{1}{2})^2)^{1/2}((\xi^2 - \tfrac{1}{2})^2)^{1/2} + 2(\tfrac{1}{4} - (\eta^2 - \tfrac{1}{2})^2)^{1/2}((\xi^2 - \tfrac{1}{2})^2)^{1/2} \leqslant \tfrac{1}{2} \quad (12)$$

which is readily verified by the Schwartz inequality. This proves that the condition

$$\Delta t \leqslant \Delta x/\rho(A) \qquad \text{and} \qquad \Delta t \leqslant \Delta y/\rho(B) \tag{13}$$

is a sufficient condition for stability.

It is interesting in this stage to check whether we lose accuracy by using (6) instead of the two-dimensional version of (2). We would like to check the phase error of (6), but unlike [1] we consider the fully discretized scheme. For (2) we get

$$\text{P.E.} = -\frac{A\alpha^3}{6\Delta x}\left[1-\left(\frac{\Delta t}{\Delta x}A\right)^2\right]-\frac{B\beta^3}{6\Delta y}\left[1-\left(\frac{\Delta t}{\Delta y}B\right)^2\right]+\frac{1}{2}\left(\alpha\frac{\Delta t}{\Delta x}A\right)\left(\beta\frac{\Delta t}{\Delta y}B\right)^2 + \frac{1}{2}\left(\frac{\Delta t}{\Delta x}\alpha A\right)^2\left(\frac{\Delta t}{\Delta y}B\beta\right) \tag{13a}$$

whereas for (6) we get

$$\text{P.E.} = -\frac{A\alpha^3}{6\Delta x}\left[1-\left(\frac{\Delta t}{\Delta x}A\right)^2\right]-\frac{B}{6\Delta y}\left[1-\left(\frac{\Delta t}{\Delta y}B\right)^2\right]-\frac{A\alpha\beta^2}{2\Delta x}\left[1-\frac{A\Delta t}{\Delta x}\left(\frac{B\Delta t}{\Delta y}\right)^2\right] - \frac{B\beta\alpha^2}{2\Delta y}\left[1-\frac{B\Delta t}{\Delta y}\left(\frac{A\Delta t}{\Delta x}\right)^2\right] \tag{13b}$$

where $\alpha = w_x \Delta x$ $\beta = w_y \Delta y$. A and B are scalars now. From (13b) it is clear that one minimizes the P.E. by taking $(\Delta t/\Delta x)\, A$ and $(\Delta t/\Delta y)\, B$ to be as close as possible to the stability limit $(\Delta t/\Delta x)\, A \sim 1$ and $(\Delta t/\Delta y)\, B \sim 1$. This does not happen in (6a) since the last two terms will grow. It is clear that (13b) presents better Phase error than (13a). This surprising fact can be explained by looking at the truncation error. For scalar linear equations, the scheme (6) is fourth-order accurate if $(\Delta t/\Delta x)\, A = 1$ and $(\Delta t/\Delta y)\, B = 1$. This is not true for (2).

For arbitrary A and B condition (13) allows a time step which is twice as large as that allowed by (4). In order to decide which scheme is more efficient we have to compare the number of operations required by the leap-frog scheme (2) and the modified one (6). Let (P_1, P_2, P_3) denote the number of multiplications, divisions and additions required to evaluate the vector $(\Delta t/\Delta x)\, F$ (or $(\Delta t/\Delta y)\, G$). Let R be the number of total operations required for a scheme to advance the solution by one time step; then for (2) we get

$R_2 = 4P_1$ multiplications $+\ 4P_2$ divisions $+\ (4P_3 + 4q)$ additions whereas for (6) we get

$R_6 = (4P_1 + 4q)$ multiplications $+\ 4P_2$ divisions $+\ (4P_3 + 8q)$ additions. For nontrivial problems $(R_6/R_2) \ll 2$ and therefore the scheme (6) is much more efficient.

The above discussion did not take into account any specific form of the vectors F and G. We would like to discuss now in more detail the Euler equations of inviscid flow. In this case F and G are given by

$$u = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}; F = \begin{bmatrix} m \\ -\frac{\gamma - 3}{2}\frac{m^2}{\rho} + (\gamma - 1)\left(E - \frac{n^2}{2\rho}\right) \\ \frac{mn}{\rho} \\ \frac{1-\gamma}{2\rho^2} m(m^2 + n^2) + \frac{\gamma m E}{\rho} \end{bmatrix},$$

$$G = \begin{bmatrix} m \\ \frac{mn}{\rho} \\ -\frac{\gamma - 3}{2}\frac{n^2}{\rho} + (\gamma - 1)\left(E - \frac{m^2}{\rho}\right) \\ \frac{1-\gamma}{2\rho^2} n(m^2 + n^2) + \frac{\gamma n E}{\rho} \end{bmatrix} \tag{14}$$

It is easy to compute that

$$P_1 = 14 \quad P_2 = 5 \quad P_3 = 4 \quad q = 4$$

and therefore

$R_2 = 56$ multiplications $+$ 20 divisions $+$ 32 additions

$R_6 = 62$ multiplications $+$ 20 divisions $+$ 48 additions.

The stability condition for the scheme (2) can be put in the form

$$\Delta t\{(u^2 + v^2)^{1/2} + c\} \leqslant 1/2^{1/2} \tag{15}$$

and for (6)

$$\Delta t\{(u^2 + v^2)^{1/2} + c\} \leqslant 2^{1/2}. \tag{16}$$

Hence (6) is much more efficient than (2).

The three-dimensional case corresponding to Eq. (2) can be written as follows:

$$\begin{aligned} u^{n+1}_{j,k,m} = u^n_{j,k,m} &+ (\Delta t/\Delta x)[F(\tilde{u}^n_{j+1,k,m}) - F(\tilde{u}^n_{j-1,h,m})] \\ &+ (\Delta t/\Delta y)[G(\tilde{u}^n_{j,k+1,m}) - G(\tilde{u}^n_{j,k-1,m})] \\ &+ (\Delta t/\Delta z)[H(\tilde{u}_{j,k,m+1}) - H(\tilde{u}_{j,k,m-1})] \end{aligned} \tag{17}$$

where

$$\tilde{u}_{j+1,k,m} = \tfrac{1}{4}[u_{j+1,k+1,m+1} + u_{j+1,k-1,m-1} + u_{j+1,k+1,m-1} + u_{j+1,k-1,m+1}]$$

The stability condition can be shown to be

$$\frac{\Delta t}{\Delta x}\,\rho(A) \leqslant \frac{3^{1/2}}{2} \approx 0.866$$
$$\frac{\Delta t}{\Delta y}\,\rho(B) \leqslant \frac{3^{1/2}}{2}\,; \qquad \frac{\Delta t}{\Delta z}\,\rho(C) \leqslant \frac{3^{1/2}}{2}\,. \tag{18}$$

It is not known if there is some other averaging that improves condition (18). It can be shown, however, that one cannot get maximum stability, i.e.,

$$(\Delta t/\Delta x)\,\rho(A) \leqslant 1 \quad (\Delta t/\Delta y)\,\rho(B) \leqslant 1 \quad (\Delta t/\Delta z)\,\rho(C) \leqslant 1. \tag{19}$$

It is our feeling that (17) is optimal in the sense that it combines simple averaging and large time steps.

REFERENCES

1. H. O. KREISS AND J. OLIGER, Method for the approximation of time dependent problems, GARP Publ. Ser. No. 10, 1973.
2. E. TURKEL, *Math. Comp.* **27** (1973), 729–736.
3. R. D. RICHTMYER AND K. W. MORTON, "Difference Methods for Initial Value Problems," Interscience, çw York, 1967.

RECEIVED: November 11, 1975.

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